GROMOV-WITTEN INVARIANTS OF THE MODULI OF BUNDLES ON A SURFACE

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1. Introduction

Let $\Sigma = \Sigma_g$ be a compact Riemann surface of genus $g \geq 2$ and let M_Σ stand for the moduli space of rank two stable vector bundles on Σ with odd (and fixed) determinant. In [5] the author produced a presentation for the quantum cohomology ring $QH^*(M_\Sigma)$ in terms of its natural generators by giving the relations satisfied by them. Here we want to show that this information yields all the multiple-point Gromov-Witten invariants on the generators, or which is equivalent, all 3-point Gromov-Witten invariants on the elements which are quantum products of the generators.

On the other hand consider the (instanton) Floer cohomology $HF^*(Y)$ of the three-manifold $Y = \Sigma \times \mathbb{S}^1$ endowed with the $SO(3)$-bundle with $w_2 = \text{P.D.}[\mathbb{S}^1] \in H^2(Y; \mathbb{Z}/2\mathbb{Z})$, which is determined in [4]. We show that the ring structure of $HF^*(Y)$ yields the Donaldson invariants $D_S^{c_1}$ of the algebraic surface $S = \Sigma \times \mathbb{P}^1$, for Kähler metrics whose period point is close enough to Σ in the Kähler cone, $U(2)$-bundles whose first Chern class $c_1 \in H^2(S, \mathbb{Z})$ satisfies $c_1 \cdot \Sigma \equiv 1 \pmod 2$, and on any collection of homology classes coming from $\Sigma \subset S$. Moreover the isomorphism $QH^*(M_\Sigma) \cong HF^*(\Sigma \times \mathbb{S}^1)$ gives an equality between these Donaldson invariants of S and the multiple-point Gromov-Witten invariants of M_Σ on the generators.

What the quantum cohomology $QH^*(M_\Sigma)$ does not give is the 3-point Gromov-Witten invariants on arbitrary homology classes (i.e. on the elements which are cup products of the generators). This is equivalent to knowing the quantum product of two arbitrary homology classes, and hence to knowing all the multiple-point Gromov-Witten invariants on homology classes. Again another equivalent formulation to this problem is to obtain the (just vector space) isomorphism

$$QH^*(M_\Sigma) \xrightarrow{\cong} H^*(M_\Sigma).$$

For the case $g = 2$ this problem is settled in [1]. Here we deal with the case $g = 3$. For this it proves necessary to write down all 3-point Gromov-Witten invariants of degree 1, which

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are computed thanks to the results in [5, section 3] and moreover we need two particular Gromov-Witten invariants of degree 2, for which we use the results of [2].

This may be viewed as an indication that (a presentation of) the quantum cohomology ring of a symplectic manifold has less information than the full collection of Gromov-Witten invariants.

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2. Multiple-point Gromov-Witten invariants of M_Σ

The general reference for this section is [5].

Generators of the cohomology of M_Σ. Consider a symplectic basis $\{\gamma_1, \ldots, \gamma_{2g}\}$ of $H_1(\Sigma; \mathbb{Z})$. Also $x \in H_0(\Sigma; \mathbb{Z})$ will stand for the class of the point. The μ-map provide natural generators $\alpha = 2\mu(\Sigma) \in H^2(M_\Sigma)$, $\psi_i = \mu(\gamma_i) \in H^3(M_\Sigma)$, $1 \le i \le 2g$, and $\beta = -4\mu(x) \in H^4(M_\Sigma)$ of the cohomology ring $H^*(M_\Sigma)$. We also put

$$\mathbb{A}(\Sigma) = \mathrm{Sym}^*(H_0(\Sigma) \oplus H_2(\Sigma)) \otimes \Lambda^* H_1(\Sigma) \stackrel{\mu}{\cong} \mathbb{C}[\alpha, \beta] \otimes \Lambda^*(\psi_1, \ldots, \psi_{2g}),$$

with grading $\deg \alpha = 2$, $\deg \psi_i = 3$ and $\deg \beta = 4$. So a basic element of $\mathbb{A}(\Sigma)$ is of the form $z = \alpha^a \beta^b \psi_{i_1} \cdots \psi_{i_r}$, $a, b \ge 0$, $1 \le i_1 < \cdots < i_r \le 2g$. The cohomology class corresponding to z under the obvious epimorphism $\pi_H : \mathbb{A}(\Sigma) \twoheadrightarrow H^*(M_\Sigma)$ will be denoted by

$$\alpha_a \beta_b \psi_{i_1} \cup \cdots \cup \psi_{i_r} = \alpha \cup \overset{(a)}{\ldots} \cup \alpha \cup \beta \cup \overset{(b)}{\ldots} \cup \beta \cup \psi_{i_1} \cup \cdots \cup \psi_{i_r},$$

and leave the notation with exponents for the quantum product (note that we cannot suppress the subindices a and/or b when they are 1).

The action of the mapping class group of Σ yields an action of the symplectic group $\mathrm{Sp}\,(2g, \mathbb{Z})$ on $\{\psi_i\}$ and the invariant part $H^*(M_\Sigma)_I$ of $H^*(M_\Sigma)$ is generated by α, β and $\gamma = -2\sum_{i=1}^g \psi_i \cup \psi_{g+i}$. Also we denote $\alpha_a \beta_b \gamma_c = \alpha \cup \overset{(a)}{\ldots} \cup \alpha \cup \beta \cup \overset{(b)}{\ldots} \cup \beta \cup \gamma \cup \overset{(c)}{\ldots} \cup \gamma$ for a typical cohomology class in $H^*(M_\Sigma)_I$.

Gromov-Witten invariants of M_Σ. Let A denote the positive generator of $\pi_2(M_\Sigma) \cong H_2(M_\Sigma; \mathbb{Z})$, i.e. $\alpha[A] = 1$. Fix $d \ge 0$ and $r \ge 3$ and let $z_i \in H^{p_i}(M_\Sigma)$, $1 \le i \le r$, be homogeneous homology classes. The r-point Gromov-Witten invariant $\Psi^{M_\Sigma}_{dA}(z_1, \ldots, z_r)$ is defined to be zero if $p_1 + \cdots + p_r \ne 6g - 6 + 4d$, and in the following manner if $p_1 + \cdots + p_r = 6g - 6 + 4d$. Consider r different points $P_1, \ldots, P_r \in \mathbb{P}^1$ and represent the Poincaré duals of z_i by generic cycles $V_{z_i} \subset M_\Sigma$. Then

$$\Psi^{M_\Sigma}_{dA}(z_1, \ldots, z_r) = \#\{f : \mathbb{P}^1 \to M_\Sigma | f \text{ is holomorphic}, f_*[\mathbb{P}^1] = dA, f(P_i) \in V_{z_i}, 1 \le i \le r\}$$

where $\#$ denotes count with signs. This is well-defined in appropriate circumstances [5]. We also put $\Psi^{M_\Sigma} = \sum_{d \ge 0} \Psi^{M_\Sigma}_{dA}$ and extend the definition to non-homogeneous z_i multi-linearly.

Quantum cohomology of M_Σ. The quantum cohomology $QH^*(M_\Sigma)$ is $H^*(M_\Sigma)$ as a vector space (so they are identified by the identity map), but the ring structure of $QH^*(M_\Sigma)$, the quantum multiplication, is a deformation of the usual cup product for $H^*(M_\Sigma)$, and it is graded only modulo 4. It is defined as follows. Let $r \geq 2$ be any integer. For cohomology classes $z_i \in H^*(M_\Sigma)$, $1 \leq i \leq r$, the quantum product of these classes is $z_1 \cdots z_r \in H^*(M_\Sigma)$ defined by

$$\langle z_1 \cdots z_r, z_{r+1} \rangle = \Psi^{M_\Sigma}(z_1, \dots, z_r, z_{r+1}), \text{ for any } z_{r+1} \in H^*(M_\Sigma).$$

Obviously, the 3-point Gromov-Witten invariant completely determines the quantum product. Therefore the 3-point Gromov-Witten invariant determines the multiple-point Gromov-Witten invariant by the simple formula

$$\Psi^{M_\Sigma}(z_1, \dots, z_r) = \Psi^{M_\Sigma}(z_1, z_2, z_3 \cdots z_r),$$

for any $z_1, \dots, z_r \in H^*(M_\Sigma)$ homology classes, $r \geq 3$.

The ring $QH^*(M_\Sigma)$ is generated by α, β and ψ_i, $1 \leq i \leq 2g$. So there is an epimorphism $\pi_{QH} : \mathbb{A}(\Sigma) \twoheadrightarrow QH^*(M_\Sigma)$. The typical quantum product is denoted as

$$\alpha^a \beta^b \psi_{i_1} \cdots \psi_{i_r} = \alpha \overset{(a)}{\cdots} \alpha\beta \overset{(b)}{\cdots} \beta\psi_{i_1} \cdots \psi_{i_r}.$$

Multiple-point Gromov-Witten invariant on generators. We define the *multiple-point Gromov-Witten invariant of M_Σ on generators* in the following way. For any $z = \alpha^a \beta^b \psi_{i_1} \cdots \psi_{i_r} \in \mathbb{A}(\Sigma)$, put

$$\tilde{\Psi}^{M_\Sigma}(z) = \Psi^{M_\Sigma}(\alpha, \overset{(a)}{\dots}, \alpha, \beta, \overset{(b)}{\dots}, \beta, \psi_{i_1}, \dots, \psi_{i_r}),$$

then extend by linearity. Another way to say this is for $z \in \mathbb{A}(\Sigma)$, $\tilde{\Psi}^{M_\Sigma}(z) = \langle \pi_{QH}\, z \rangle_{M_\Sigma}$, where $\langle \cdot \rangle_{M_\Sigma} : H^*(M_\Sigma) \to \mathbb{C}$ is the pairing with the volume class. Note that $\tilde{\Psi}^{M_\Sigma}$ is invariant under the action of $\mathrm{Sp}\,(2g, \mathbb{Z})$, so it is determined by its effect on invariant elements $z \in \mathbb{A}(\Sigma)_I$. For $g \geq 3$, $\gamma = -2 \sum_{i=1}^g \psi_i \psi_{g+i} \in QH^*(M_\Sigma)$, by [5, lemma 14], so for $z = \alpha^a \beta^b \gamma^c \in \mathbb{A}(\Sigma)$ we have

$$\tilde{\Psi}^{M_\Sigma}(z) = \Psi^{M_\Sigma}(\alpha, \overset{(a)}{\dots}, \alpha, \beta, \overset{(b)}{\dots}, \beta, \gamma, \overset{(a)}{\dots}, \gamma).$$

$\tilde{\Psi}^{M_\Sigma}$ determines the quantum cohomology $QH^*(M_\Sigma)$. Actually $QH^*(M_\Sigma) = \mathbb{A}(\Sigma)/I$, for an ideal $I \subset \mathbb{A}(\Sigma)$ of relations. Then $I = \{R \in \mathbb{A}(\Sigma) | \tilde{\Psi}^{M_\Sigma}(Rz) = 0, \forall z \in \mathbb{A}(\Sigma)\}$. The following result is a converse of this.

Proposition 1. *The quantum cohomology of M_Σ and the value of $\langle \gamma_{g-1} \rangle_{M_\Sigma} = 2^{g-1} g!$ determines the multiple-point Gromov-Witten invariant on generators $\tilde{\Psi}^{M_\Sigma}$.*

Proof. Let $z \in \mathbb{A}(\Sigma)$. As $\tilde{\Psi}^{M_\Sigma}(z)$ is invariant under the action of $\mathrm{Sp}\,(2g, \mathbb{Z})$, we may project z to the invariant part $\mathbb{A}(\Sigma)_I \subset \mathbb{A}(\Sigma)$, i.e. we may suppose that z is invariant. A basis for $QH^*(M_\Sigma)_I$ is $\alpha^a \beta^b \gamma^c$, $a + b + c < g$, by [5, section 5]. This means that $z = R +$

$\sum_{a+b+c<g} c_{abc}\alpha^a\beta^b\gamma^c$, for some numbers $c_{abc} \in \mathbb{C}$ and a relation $R \in \mathbb{A}(\Sigma)_I$ for $QH^*(M_\Sigma)_I$. By definition of relation we have that $\tilde{\Psi}^{M_\Sigma}(R) = 0$. Also $\tilde{\Psi}^{M_\Sigma}(\alpha^a\beta^b\gamma^c) = 0$, for $a+b+c<g$ and $(a,b,c) \neq (0,0,g-1)$, since in that case $\deg(\alpha^a\beta^b\gamma^c) = 2a+4b+6c < 6g-6$. Hence $\tilde{\Psi}^{M_\Sigma}(z) = c_{0,0,g-1}\tilde{\Psi}^{M_\Sigma}(\gamma^{g-1})$. Finally the value of $\tilde{\Psi}^{M_\Sigma}(\gamma^{g-1}) = \langle \gamma_{g-1}, [M_\Sigma]\rangle = \langle \gamma_{g-1}\rangle_{M_\Sigma} = 2^{g-1}g!$, by [7]. □

Let $z_i = \alpha^{a_i}\beta^{b_i}\psi_{i,j_1}, \dots, \psi_{i,j_{r_i}}$, $i \in I$, be a collection of elements of $\mathbb{A}(\Sigma)$ such that $\mathcal{B}_H = \{\pi_H z_i\}_{i\in I}$ is a basis for $H^*(M_\Sigma)$. Then $\mathcal{B}_{QH} = \{\pi_{QH} z_i\}_{i\in I}$ will be a basis for $QH^*(M_\Sigma)$. The following statements are equivalent

(1) We have a presentation of $QH^*(M_\Sigma)$, i.e. we know the ideal $I \subset \mathbb{A}(\Sigma)$ such that $QH^*(M_\Sigma) = \mathbb{A}(\Sigma)/I$.
(2) We know how to compute $\tilde{\Psi}^{M_\Sigma}(z)$, for any $z \in \mathbb{A}(\Sigma)$.
(3) We know the 3-point Gromov-Witten invariant on quantum products of the generators, i.e. $\Psi^{M_\Sigma}(\pi_{QH}z_i, \pi_{QH}z_j, \pi_{QH}z_k)$, for all $i,j,k \in I$.
(4) We know the coefficients c_{ijk} such that $\pi_{QH}z_i\,\pi_{QH}z_j = \sum_k c_{ijk}\pi_{QH}z_k$, for $i,j \in I$.

The equivalence of 1 and 2 is proposition 1. The equivalence of 2 and 3 follows from $\Psi^{M_\Sigma}(\pi_{QH}z_i, \pi_{QH}z_j, \pi_{QH}z_k) = \tilde{\Psi}^{M_\Sigma}(z_iz_jz_k)$. The equivalence of 3 and 4 follows from the definition (using the intersection pairing). The statement 1 is true because of [5], so the other three follow, meaning at least that we can perform all computations for any fixed genus g by hand or with a computer.

A full knowledge of the Gromov-Witten invariants requires to know the 3-point Gromov-Witten invariants on homology classes $\pi_H z_i = \alpha_{a_i}\beta_{b_i}\psi_{i,j_1} \cup \cdots \cup \psi_{i,j_{r_i}}$, $i \in I$. The following statements are equivalent

(1) We know the 3-point Gromov-Witten invariant on arbitrary homology classes, i.e. $\Psi^{M_\Sigma}(\pi_H z_i, \pi_H z_j, \pi_H z_k)$, for any $i,j,k \in I$.
(2) We know the coefficients d_{ijk} such that $\pi_H z_i\,\pi_H z_j = \sum_k d_{ijk}\pi_H z_k$, for any $i,j \in I$.
(3) We know the isomorphism $H^*(M_\Sigma) \xrightarrow{\cong} QH^*(M_\Sigma)$ in terms of the basis $\mathcal{B}_H$ and $\mathcal{B}_{QH}$, respectively.

The equivalence of 1 and 2 is obvious using the intersection pairing. If we have 3 then 2 follows by translating the point 4 above (which we already have) through the isomorphism. Conversely, if we have 2 we may quantum multiply the generators α, β and ψ_i, $1 \le i \le 2g$, repeteadly until we get any $\pi_{QH}z_i$ in terms of the basis $\mathcal{B}_H$. All this information is stronger than a presentation of the quantum cohomology ring, as we shall see in the example of section 4.

3. Donaldson invariants of $S = \Sigma \times \mathbb{P}^1$

The general reference for this section is [4].

Donaldson invariants of S. Consider the algebraic surface $S = \Sigma \times \mathbb{P}^1$. In particular S is a smooth 4-manifold with $b_1 = 2g$ and $b^+ = 1$. The Donaldson invariants of S depend on a metric as explained in [3]. The Kähler cone of S is $\mathcal{K} = \{a\Sigma + b\mathbb{P}^1 | a, b > 0\} \subset H^2(S)$. We set $\mathbb{A}(S) = \mathrm{Sym}^*(H_0(S) \oplus H_2(S)) \otimes \Lambda^* H_1(S)$, which is graded giving degree $4 - i$ to the classes in $H_i(S)$. The class of the point will be denoted by $x \in H_0(S)$. The Donaldson invariants are linear functionals

$$D^{c_1}_{S,H} : \mathbb{A}(S) \to \mathbb{C},$$

depending on the first Chern class $c_1 \in H^2(S;\mathbb{Z})$ of the $U(2)$-bundles involved, and on a (generic) polarisation $H \in \mathcal{K}$.

We shall consider only polarisations close to Σ, i.e. $H = \Sigma + \varepsilon \mathbb{P}^1$, with $\varepsilon > 0$ small (how small depending on the degree of the element on which we are computing the Donaldson invariants). So $D^{c_1}_S$ shall stand for $D^{c_1}_{S,H}$ with such polarisation H. Also we only consider the cases with $c_1 \cdot \Sigma$ odd. As $D^{c_1+2\alpha}_S = (-1)^{\alpha^2} D^{c_1}_S = D^{c_1}_S$, for $\alpha \in H^2(S;\mathbb{Z})$, we may suppose that $c_1 = \mathbb{P}^1$ or $c_1 = \mathbb{P}^1 + \Sigma$. If $c_1 = \mathbb{P}^1$ then $D^{c_1}_S$ is non-zero only on elements of degree $6g - 6 + 4d$, $d \geq 0$ even. If $c_1 = \mathbb{P}^1 + \Sigma$ then $D^{c_1}_S$ is non-zero only on elements of degree $6g - 6 + 4d$, $d \geq 0$ odd. We collect all these invariants together by putting $w = \mathbb{P}^1$ and $D^{(w,\Sigma)}_S = D^w_S + D^{w+\Sigma}_S$.

A basic element $z \in \mathbb{A}(S)$ is of the form $z = \Sigma^a x^b \gamma_{i_1} \cdots \gamma_{i_r} (\mathbb{P}^1)^e$. We restrict to classes coming from $\Sigma \subset S$, i.e. $z \in \mathbb{A}(\Sigma) \subset \mathbb{A}(S)$. Therefore $e = 0$ and $z = \Sigma^a x^b \gamma_{i_1} \cdots \gamma_{i_r}$. As in section 2 we identify $\mathbb{A}(\Sigma) \cong \mathbb{C}[\alpha, \beta] \otimes \Lambda^*(\psi_1, \ldots, \psi_{2g})$ by $\alpha = 2\Sigma$, $\beta = -4x$ and $\psi_i = \gamma_i$, $1 \leq i \leq 2g$. Also we let $\gamma = -2\sum \psi_i \psi_{g+i}$.

Floer cohomology of $\Sigma \times \mathbb{S}^1$. Consider the three-manifold $Y = \Sigma \times \mathbb{S}^1$ and let $HF^*(Y)$ stand for its (instanton) Floer cohomology endowed with the $SO(3)$-bundle with $w_2 =$ P.D.$[\mathbb{S}^1]$. By the analysis in [4], $HF^*(Y)$ is generated by α, β, ψ_i, $1 \leq i \leq 2g$, coming from the relative Donaldson invariants for $\Sigma \times D^2$ of 2Σ, $-4x$, γ_i, $1 \leq i \leq 2g$, respectively.

There is an evaluation map [4, corollary 19] $\langle \cdot \rangle_g : HF^*(Y) \to \mathbb{C}$ with the property that for any $z = \alpha^a \beta^b \psi_{i_1} \cdots \psi_{i_r} \in \mathbb{A}(\Sigma)$, we have

$$D^{(w,\Sigma)}_S(z) = D^{(w,\Sigma)}_S((2\Sigma)^a (-4x)^b \gamma_{i_1} \cdots \gamma_{i_r}) = \langle \alpha^a \beta^b \psi_{i_1} \cdots \psi_{i_r} \rangle_g,$$

The Donaldson invariants $D^{(w,\Sigma)}_S$ determine the Floer homology since $HF^*(Y) = \mathbb{A}(\Sigma)/J$ and $J = \{R \in \mathbb{A}(\Sigma) | D^{(w,\Sigma)}_S(Rz) = 0, \forall z \in \mathbb{A}(\Sigma)\}$. The following result is a converse of this.

Proposition 2. *The Floer cohomology of $Y = \Sigma \times \mathbb{S}^1$ and the value $D^w_S(\gamma^{g-1}) = -2^{g-1} g!$ determine the Donaldson invariants $D^{(w,\Sigma)}_S(z)$, for any $z \in \mathbb{A}(\Sigma)$.*

Proof. The argument is similar to that of proposition 1. We may take $z \in \mathbb{A}(\Sigma)$ invariant under $\mathrm{Sp}\,(2g, \mathbb{Z})$. A vector basis of $HF^*(Y)_I$ is $\alpha^a \beta^b \gamma^c$, $a + b + c < g$, by [4]. Then $z = R + \sum_{a+b+c<g} c_{abc} \alpha^a \beta^b \gamma^c$, for some $c_{abc} \in \mathbb{C}$ and $R \in \mathbb{A}(\Sigma)_I$ which is a relation for $HF^*(Y)_I$.

Then $D_S^{(w,\Sigma)}(R)=0$ and $D_S^{(w,\Sigma)}(\alpha^a\beta^b\gamma^c)=0$, for $a+b+c<g$ and $(a,b,c)\neq(0,0,g-1)$, since in that case $\deg(\alpha^a\beta^b\gamma^c)=2a+4b+6c<6g-6$ and there is no moduli space of ASD connections of dimension smaller than $6g-6$. Hence $D_S^{(w,\Sigma)}(z)=c_{0,0,g-1}D_S^w(\gamma^{g-1})=-c_{0,0,g-1}\langle\gamma_{g-1},[M_\Sigma]\rangle$. □

Theorem 3. *Suppose that $g\geq 3$. Then for any $z=\alpha^a\beta^b\psi_{i_1}\cdots\psi_{i_r}\in\mathbb{A}(\Sigma)$ with $\deg z=6g-6+4d$, $d\geq 0$, we have*

$$\Psi_{dA}^{M_\Sigma}(\alpha,\overset{(a)}{\ldots},\alpha,\beta,\overset{(b)}{\ldots},\beta,\psi_{i_1},\ldots,\psi_{i_r})=(-1)^{gd+1}D_S^{(w,\Sigma)}((2\Sigma)^a(-4x)^b\gamma_{i_1}\cdots\gamma_{i_r}).$$

Proof. By [5, corollary 21], for any $g\geq 3$ there is an isomorphism $QH^*(M_\Sigma)\xrightarrow{\cong}HF^*(Y)$, $(\alpha,\beta,\psi_i)\mapsto(\varepsilon^{2g}\alpha,\varepsilon^{4g}\beta,\varepsilon^{3g}\psi_i)$, where ε is a primitive eighth root of unity. As $D_S^w(\gamma^{g-1})=-\tilde{\Psi}^{M_\Sigma}(\gamma^{g-1})$ the result follows easily. □

Corollary 4. *Suppose that $g\geq 4$. Then for any $z=\alpha^a\beta^b\psi_{i_1}\cdots\psi_{i_r}\in\mathbb{A}(\Sigma)$ with $\deg z=6(g-1)-6+4d$, $d\geq 0$, we have*

$$\Psi_{dA}^{M_{\Sigma_g}}(\gamma,\alpha,\overset{(a)}{\ldots},\alpha,\beta,\overset{(b)}{\ldots},\beta,\psi_{i_1},\ldots,\psi_{i_r})=(-1)^d 2g\Psi_{dA}^{M_{\Sigma_{g-1}}}(\alpha,\overset{(a)}{\ldots},\alpha,\beta,\overset{(b)}{\ldots},\beta,\psi_{i_1},\ldots,\psi_{i_r}).$$

Proof. By [4, corollary 19] it is $D_{\Sigma_g\times\mathbb{P}^1}^{(w,\Sigma)}(\gamma z)=2gD_{\Sigma_{g-1}\times\mathbb{P}^1}^{(w,\Sigma)}(z)$, for any $z\in\mathbb{A}(\Sigma)$. Then the result follows from theorem 3. □

Generating function. Proposition 2 can be used for a given genus g to effectively compute the Donaldson invariants $D_S^{(w,\Sigma)}$ on homology classes coming from $\Sigma\subset S$. For this, we collect the Donaldson invariants into a generating function as follows (we only need the invariant part)

$$D_S^{(w,\Sigma)}(e^{s\alpha+\lambda\beta+r\gamma})=\langle e^{s\alpha+\lambda\beta+r\gamma}\rangle_g. \tag{1}$$

Every relation $R(\alpha,\beta,\gamma)$ for $HF^*(Y)_I$ gives a differential equation $R(\frac{\partial}{\partial s},\frac{\partial}{\partial\lambda},\frac{\partial}{\partial r})$ satisfied by (1). Actually (1) is the only solution $F(s,\lambda,r)$ to all these differential equations satisfying the initial conditions

$$\begin{cases}\frac{\partial^{a+b+c}F}{\partial s^a\partial\lambda^b\partial r^c}\Big|_{s=\lambda=r=0}=0, & a+b+c<g,\ (a,b,c)\neq(0,0,g-1)\\ \frac{\partial^{g-1}F}{\partial r^{g-1}}\Big|_{s=\lambda=r=0}=-2^{g-1}g! & \end{cases}$$

The decomposition $HF^*(Y)_I=\bigoplus_{r=-(g-1)}^{g-1}R_{g,r}$ in [4, section 7] may be useful for finding $F(s,\lambda,r)$. The map $\langle\cdot\rangle_g:HF^*(Y)_I\to\mathbb{C}$ yields maps $\langle\cdot\rangle_{g,r}:R_{g,r}\to\mathbb{C}$ such that $\langle\cdot\rangle_g=\sum_{r=-(g-1)}^{g-1}\langle\cdot\rangle_{g,r}$. Now let $\Phi_r=\langle e^{s\alpha+\lambda\beta+r\gamma}\rangle_{g,r}$. Then the relations for $R_{g,r}$ give the differential equations satisfied by every Φ_r and $D_S^{(w,\Sigma)}(e^{s\alpha+\lambda\beta+r\gamma})=\sum_r\Phi_r$.

Example 5. Let the genus $g = 2$. By [4, example 22]

$$HF^*(Y)_I = R_{2,-1} \oplus R_{2,0} \oplus R_{2,1} = \frac{\mathbb{C}[\alpha,\beta,\gamma]}{(\alpha-4,\beta+8,\gamma)} \oplus \frac{\mathbb{C}[\alpha,\beta,\gamma]}{(\alpha^2,\beta-8,\gamma+16\alpha)} \oplus \frac{\mathbb{C}[\alpha,\beta,\gamma]}{(\alpha+4,\beta+8,\gamma)}.$$

Associated to $R_{2,-1}$ we have the differential equations $\frac{\partial}{\partial s}-4$, $\frac{\partial}{\partial \lambda}+8$ and $\frac{\partial}{\partial r}$, whose solution is a multiple of $e^{4s-8\lambda}$. For $R_{2,0}$ we have the differential equations $\frac{\partial^2}{\partial s^2}$, $\frac{\partial}{\partial \lambda}-8$ and $\frac{\partial}{\partial r}+16\frac{\partial}{\partial s}$, whose solutions are linear combinations of $e^{8\lambda}$ and $(16r-s)e^{8\lambda}$. Finally for $R_{2,1}$ we get $e^{-4s-8\lambda}$. Then $D_S^{(w,\Sigma)}(e^{s\alpha+\lambda\beta+r\gamma})$ is a linear combination of $e^{4s-8\lambda}$, $e^{8\lambda}$, $(16r-s)e^{8\lambda}$ and $e^{-4s-8\lambda}$. Putting the initial conditions one gets that

$$D_S^{(w,\Sigma)}(e^{s\alpha+\lambda\beta+r\gamma}) = -\frac{1}{16}\sinh 4s\, e^{-8\lambda} - \frac{1}{4}(16r-s)e^{8\lambda}.$$

Example 6. In the case of genus $g = 3$ we use the decomposition in [4, example 23] to obtain that

$$D_S^{(w,\Sigma)}(e^{s\alpha+\lambda\beta+r\gamma}) = \frac{1}{2048}\cos 8s\, e^{8\lambda} + \frac{3}{128}\cosh 4s\, e^{-8\lambda}$$
$$-\frac{1}{32}(s\sinh 4s - 4\lambda\cosh 4s + 12r\sinh 4s)e^{-8\lambda} - \left(\frac{49}{2048} - \frac{1}{4}\lambda + \frac{3}{64}(16r-s)^2 + \lambda^2\right)e^{8\lambda}$$

This agrees with [6, formula (5.16)].

Examples 5 and 6 agree with the fact that $\frac{\partial}{\partial r} D^{(w,\Sigma)}_{\Sigma_g\times\mathbb{P}^1}(e^{s\alpha+\lambda\beta+r\gamma}) = \langle \gamma e^{s\alpha+\lambda\beta+r\gamma}\rangle_g = 2g\langle e^{s\alpha+\lambda\beta+r\gamma}\rangle_{g-1} = 2g D^{(w,\Sigma)}_{\Sigma_{g-1}\times\mathbb{P}^1}(e^{s\alpha+\lambda\beta+r\gamma})$. Note also that by theorem 3,

$$\tilde{\Psi}^{M_\Sigma}(e^{s\alpha+\lambda\beta+r\gamma}) = \begin{cases} (-1)^{\frac{g+1}{2}} D_S^{(w,\Sigma)}(e^{is\alpha-\lambda\beta-ir\gamma}) & g \text{ odd} \\ -D_S^{(w,\Sigma)}(e^{s\alpha+\lambda\beta+r\gamma}) & g \text{ even} \end{cases}$$

4. Gromov-Witten invariants in the case $g = 3$

In this section we shall compute all the Gromov-Witten invariants for the moduli space M_Σ of rank 2 odd degree (and fixed determinant) stable bundles over the Riemann surface of genus $g = 3$. The dimension of M_Σ is $\dim M_\Sigma = 6g-6 = 12$ and its cohomology ring is written in [5, proposition 1] to be

$$H^*(M_\Sigma) = \frac{\mathbb{C}[\alpha,\beta,\gamma]}{(q_3^1,q_3^2,q_3^3)} \oplus \left(H^3 \otimes \frac{\mathbb{C}[\alpha,\beta,\gamma]}{(q_2^1,q_2^2,q_2^3)}\right) \oplus \left(\Lambda_0^2 H^3 \otimes \frac{\mathbb{C}[\alpha,\beta,\gamma]}{(q_1^1,q_1^2,q_1^3)}\right), \tag{2}$$

where $H^3 = \langle \psi_1,\dots,\psi_6\rangle \cong H_1(\Sigma)$, $\Lambda_0^2 H^3 = \langle \psi_i \cup \psi_j | 1 \le i < j \le 6, j \ne i+3\rangle \oplus \langle \psi_1\cup\psi_4 - \psi_2\cup\psi_5, \psi_1\cup\psi_4 - \psi_3\cup\psi_6\rangle$ and

$$\begin{array}{lll} q_1^1 = \alpha, & q_2^1 = \alpha_2 + \beta, & q_3^1 = \alpha_3 + 5\alpha_1\beta_1 + 4\gamma, \\ q_1^2 = \beta, & q_2^2 = \alpha_1\beta_1 + \gamma, & q_3^2 = \alpha_2\beta_1 + \beta_2 + \frac{4}{3}\gamma_1\alpha_1, \\ q_1^3 = \gamma, & q_2^3 = \gamma_1\alpha_1, & q_3^3 = \gamma_1\alpha_2 + \gamma_1\beta_1. \end{array}$$

The quantum cohomology is [5, theorem 20]

$$QH^*(M_\Sigma) = \frac{\mathbb{C}[\alpha,\beta,\gamma]}{(Q^1_3,Q^2_3,Q^3_3)} \oplus \left(H^3 \otimes \frac{\mathbb{C}[\alpha,\beta,\gamma]}{(Q^1_2,Q^2_2,Q^3_2)}\right) \oplus \left(\Lambda^2_0 H^3 \otimes \frac{\mathbb{C}[\alpha,\beta,\gamma]}{(Q^1_1,Q^2_1,Q^3_1)}\right), \tag{3}$$

where $\Lambda^2_0 H^3 = \langle \psi_i\psi_j | 1 \leq i < j \leq 6, j \neq i+3\rangle \oplus \langle \psi_1\psi_4 - \psi_2\psi_5, \psi_1\psi_4 - \psi_3\psi_6\rangle$ and

$$\begin{array}{lll} Q^1_1 = \alpha, & Q^1_2 = \alpha^2+\beta+8, & Q^1_3 = \alpha^3 + 5\alpha\beta + 4\gamma - 24\alpha, \\ Q^2_1 = \beta+8, & Q^2_2 = \alpha\beta+\gamma-8\alpha, & Q^2_3 = \alpha^2\beta + \beta^2 + \frac{4}{3}\gamma\alpha + 8\alpha^2 + 16\beta + 64, \\ Q^3_1 = \gamma, & Q^3_2 = \gamma\alpha, & Q^3_3 = \gamma\alpha^2 + \gamma\beta + 8\gamma. \end{array}$$

We shall determine the isomorphism $H^*(M_\Sigma) \cong QH^*(M_\Sigma)$, which is equivalent to the Gromov-Witten invariants by the discussion at the end of section 2. We remark that the three pieces of the decompositions (2) and (3) correspond by the proof of [5, proposition 16].

The invariant part. A basis for $H^*(M_\Sigma)_I$ is $\mathcal{B}_H = \{1,\alpha,\beta,\alpha_2,\gamma,\alpha_1\beta_1,\beta_2,\alpha_1\gamma_1,\beta_1\gamma_1,\gamma_2\}$ and a basis for $QH^*(M_\Sigma)_I$ is $\mathcal{B}_{QH} = \{1,\alpha,\beta,\alpha^2,\gamma,\alpha\beta,\beta^2,\alpha\gamma,\beta\gamma,\gamma^2\}$. The isomorphism $H^*(M_\Sigma)_I \cong QH^*(M_\Sigma)_I$ is given by

$$\left\{ \begin{array}{lllllllll} \alpha^2 & = & \alpha_2 & + & A_1 & & & & \\ \alpha\beta & = & \alpha_1\beta_1 & + & A_2\alpha & & & & \\ \beta^2 & = & \beta_2 & + & A_3\beta + A_4\alpha_2 & + & B_1 & & \\ \alpha\gamma & = & \alpha_1\gamma_1 & + & A_5\beta + A_6\alpha_2 & + & B_2 & & \\ \beta\gamma & = & \beta_1\gamma_1 & + & A_7\gamma + A_8\alpha_1\beta_1 & + & B_3\alpha & & \\ \gamma^2 & = & \gamma_2 & + & A_9\gamma_1\alpha_1 + A_{10}\beta_2 & + & B_4\alpha_2 + B_5\beta & + & C \end{array} \right.$$

for constants $A_1,\dots,A_{10}$, $B_1,\dots,B_5$ and C, determined by the Gromov-Witten invariants of degree 1, 2 and 3, respectively. Now we shall compute these constants. To start with we write the top-products from [7] (alternatively we may use $\langle\gamma_2\rangle_{M_\Sigma} = 24$ and the relations for $H^*(M_\Sigma)_I$)

$$\begin{array}{llll} \langle\alpha_6\rangle_{M_\Sigma} = 7\cdot 32, & \langle\alpha_4\beta_1\rangle_{M_\Sigma} = -64, & \langle\alpha_2\beta_2\rangle_{M_\Sigma} = 32, & \langle\beta_3\rangle_{M_\Sigma} = 0, \\ \langle\alpha_3\gamma_1\rangle_{M_\Sigma} = 24, & \langle\alpha_1\beta_1\gamma_1\rangle_{M_\Sigma} = -24 & \text{and} \quad \langle\gamma_2\rangle_{M_\Sigma} = 24. & \end{array}$$

The Gromov-Witten invariants of degree one. Let us recall the space N given in [5, section 3], which parametrises non-split extensions on Σ of the form

$$0 \to L \to E \to \Lambda \otimes L^{-1} \to 0,$$

where L is a line bundle of degree zero. Let J be the Jacobian variety parametrising line bundles of degree zero on Σ. The natural isomorphism $H_1(\Sigma) \cong H^1(J)$ associates to $\{\gamma_i\}$ a symplectic basis $\{\phi_i\}$ of $H^1(J)$. Put $\omega = \sum_{i=1}^3 \phi_i \wedge \phi_{3+i}$. Then N sits as a fibration $\mathbb{P}^2 \to N = \mathbb{P}(\mathcal{E}^\vee) \to J$, where $\mathcal{E}$ is a rank-3 bundle over N with Chern classes $c_i = c_i(\mathcal{E}) = (4^i/i!)\omega^i$. The cohomology ring of N is generated by the classes $\phi_i \in H^1(N)$,

$1 \leq i \leq 6$ and the class of the hyperplane $h \in H^2(N)$. Actually $H^*(N)$ is freely generated (in a graded sense) subject to the only relation

$$h^3 + c_1 h^2 + c_2 h + c_3 = 0.$$

(We only use cup-products in $H^*(N)$, so we write $\phi_{i_1} \cdots \phi_{i_r} h^j = \phi_{i_1} \cup \cdots \cup \phi_{i_r} \cup h \cup \overset{(j)}{\ldots} \cup h$.) There is an action of the mapping class group of Σ on N, which gives an action of $\mathrm{Sp}\,(2g, \mathbb{Z})$ on $\{\phi_i\}$ and the invariant part is generated by ω and h.

There is a natural map $i : N \to M_\Sigma$. We need to write down how our basis for $H^*(M_\Sigma)_I$ restrict to $H^*(N)$. By [5, equation (12)] $\alpha = 4\omega + h$, $\beta = h^2$, $\psi_i = -h\phi_i$, $1 \leq i \leq 6$, and $\gamma = -2\omega h^2$ in N (we omit i^* as there is no danger of confussion). Therefore

$$\begin{cases} \alpha = 4\omega + h \\ \beta = h^2 \\ \alpha_2 = 16\omega^2 + 8\omega h + h^2 \\ \gamma = -2\omega h^2 \\ \alpha_1\beta_1 = 4\omega h^2 + h^3 = -8\omega^2 h - \frac{32}{3}\omega^3 \\ \beta_2 = h^4 = 8\omega^2 h^2 + \frac{64}{3}\omega^3 h \\ \alpha_1\gamma_1 = -8\omega^2 h^2 - 2\omega h^3 = 16\omega^3 h \\ \beta_1\gamma_1 = -2\omega h^3 = -16\omega^3 h^2 \\ \gamma_2 = 0 \end{cases}$$

Let $l \in H_2(N; \mathbb{Z})$ be the class of the line in a fibre $\mathbb{P}^2 \subset N$. Then $i_* l = A$. The Gromov-Witten invariants of degree 1 of N and M_Σ satisfy $\Phi_l^N = \Phi_A^{M_\Sigma}$ by [5, lemma 8]. Moreover the Gromov-Witten invariants of N are computed as in [5, lemma 10]. Let $z_i = \omega^{a_i} h^{b_i} \in H^*(N)$, $1 \leq i \leq 3$, $0 \leq a_i \leq 3$, $0 \leq b_i \leq 2$. Put $a = a_1 + a_2 + a_3$ and $b = b_1 + b_2 + b_3$ and suppose $2a + 2b = 6g - 2 = 16$. Then

$$\Psi_l^N(z_1, z_2, z_3) = \begin{cases} \langle \omega^a X^b, [J] \rangle = \frac{(-8)^{b-5}}{(b-5)!}\omega^3 = 6\frac{(-8)^{b-5}}{(b-5)!}, & b \geq 5 \\ 0, & b < 5 \end{cases}$$

where $X^{2g-1+i} = X^{5+i} = ((-8)^i/i!)\omega^i$ as in [5, lemma 10]. With this we may find the coefficients $A_1, \ldots, A_{10}$. For instance we have

$$\left.\begin{array}{l} \Psi_A^{M_\Sigma}(\alpha, \alpha, \gamma_2) = \langle \alpha^2, \gamma_2 \rangle = \langle A_1, \gamma_2 \rangle = 24A_1 \\ \Psi_l^N(\alpha, \alpha, \gamma_2) = \Psi_l^N(\omega + h, \omega + h, 0) = 0 \end{array}\right\} \implies A_1 = 0,$$

$$\left.\begin{array}{l} \Psi_A^{M_\Sigma}(\alpha, \beta, \beta_1\gamma_1) = \langle \alpha\beta, \beta_1\gamma_1 \rangle = \langle A_2\alpha, \beta_1\gamma_1 \rangle = -24A_2 \\ \Psi_l^N(\alpha, \beta, \beta_1\gamma_1) = \Psi_l^N(4\omega + h, h^2, -16\omega^3 h^2) = -16 \cdot 6 \end{array}\right\} \implies A_2 = 4.$$

Analogously we get $A_3 = -12$, $A_4 = -8$, $A_5 = -3$, $A_6 = -3$, $A_7 = -20$, $A_8 = -12$, $A_9 = 8$ and $A_{10} = -6$.

More constants from pairings. Once we know the coefficients $A_1, \dots, A_{10}$, we may get some relations between the coefficients $B_1, \dots, B_5$ and C by working out the pairings of the elements of the basis $\mathcal{B}_{QH}$ written in terms of those of $\mathcal{B}_H$. The only pairings in which the coefficients $B_1, \dots, B_5$ and C are going to appear are those between elements whose degrees add up to $6g-6+4d$, $d \geq 2$, i.e. $\langle \beta\gamma, \beta\gamma \rangle$, $\langle \gamma^2, \beta^2 \rangle$, $\langle \gamma^2, \gamma\alpha \rangle$ and $\langle \gamma^2, \gamma^2 \rangle$. The relations in $QH^*(M_\Sigma)_I$ imply that $\gamma^3 = 0$ and $\gamma^2(\beta^2 - 64) = 0$, so $\langle \beta\gamma, \beta\gamma \rangle = \langle \beta^2\gamma^2 \rangle_{M_\Sigma} = \langle 64\gamma^2 \rangle_{M_\Sigma} = 64 \cdot 24$, $\langle \gamma^2, \beta^2 \rangle = 64 \cdot 24$, $\langle \gamma^2, \gamma\alpha \rangle = 0$ and $\langle \gamma^2, \gamma^2 \rangle = 0$. So we get

$$64 \cdot 24 = \langle \beta\gamma, \beta\gamma \rangle = \langle \beta_1\gamma_1 - 20\gamma - 12\alpha_1\beta_1 + B_3\alpha, \beta_1\gamma_1 - 20\gamma - 12\alpha_1\beta_1 + B_3\alpha \rangle =$$

$$= 2B_3 \langle \alpha_1\beta_1\gamma_1 \rangle_{M_\Sigma} + \langle -20\gamma - 12\alpha_1\beta_1, -20\gamma - 12\alpha_1\beta_1 \rangle = -24 \cdot 2B_3 + 24 \cdot 112,$$

and hence $B_3 = 24$. From the other pairings we obtain the equations $B_2 + B_4 - B_5 = -24$, $B_1 + \frac{4}{3}B_4 = -32$ and $C = 8B_5$, respectively. No more information can be extracted from the intersection pairing.

To finish we only need to find

$$\begin{cases} 24B_1 = \Psi_{2A}^{M_\Sigma}(\beta, \beta, \mathrm{pt}) \\ 24B_2 = \Psi_{2A}^{M_\Sigma}(\alpha, \gamma, \mathrm{pt}) \end{cases}$$

This correspond to the computation of two particular Gromov-Witten invariants of degree two, task that will be carried out in section 5. The answer is given in equations (7) and (8). We get $B_1 = 0$, $B_2 = -1$ and then $B_5 = -1$, $B_4 = -24$ and $C = -8$.

Non-invariant part. We recall that the decompositions (2) and (3) correspond since every piece is the isogeneous piece corresponding to an irreducible representation of $\mathrm{Sp}\,(2g, \mathbb{Z})$. First we shall deal with the piece corresponding to H^3. A vector basis given by the usual cohomology is $\mathcal{B}_H = \{q_i, \psi_i\alpha_1, \psi_i\beta_1, \psi_i\gamma_1 | 1 \leq i \leq 6\}$ and the basis given by the quantum cohomology is $\mathcal{B}_{QH} = \{q_i, \psi_i\alpha, \psi_i\beta, \psi_i\gamma | 1 \leq i \leq 6\}$. We must have

$$\begin{cases} \psi_i &= \psi_i \\ \psi_i\alpha &= \psi_i\alpha_1 \\ \psi_i\beta &= \psi_i\beta_1 + A_1\psi_i \\ \psi_i\gamma &= \psi_i\gamma_1 + A_2\psi_i\alpha_1 \end{cases}$$

for $1 \leq i \leq 6$ and constants A_1, A_2. We may compute these constants as above. First we need the intersection pairings

$$\langle \psi_1 \cup \psi_4\alpha_3 \rangle_{M_\Sigma} = -4, \qquad \langle \psi_1 \cup \psi_4\alpha_1\beta_1 \rangle_{M_\Sigma} = 4, \qquad \langle \psi_1 \cup \psi_4\gamma_1 \rangle_{M_\Sigma} = -4.$$

The restriction of the cohomology classes involved to N are as follows

$$\begin{cases} \psi_i = -\phi_i h \\ \psi_i\alpha_1 = -4\phi_i\omega h - \phi_i h^2 \\ \psi_i\beta_1 = -\phi_i h^3 = 4\phi_i\omega h^2 + 8\phi_i\omega^2 h \\ \psi_i\gamma_1 = 2\phi_i\omega h^3 = -8\phi_i\omega^2 h^2 \end{cases}$$

And the Gromov-Witten invariants of N that we need are

$$\Psi_l^N(\phi_1 z_1, \phi_4 z_2, z_3) = \begin{cases} \frac{(-8)^{b-5}}{(b-5)!}\phi_1\phi_2\omega^2 = 2\frac{(-8)^{b-5}}{(b-5)!}, & b \geq 5 \\ 0, & b < 5 \end{cases}$$

for $z_i = \omega^{a_i} h^{b_i} \in H^*(N)$, $1 \leq i \leq 3$, $0 \leq a_i \leq 3$, $0 \leq b_i \leq 2$, $a = a_1 + a_2 + a_3$, $b = b_1 + b_2 + b_3$ and $a + b = 7$. Then we get

$$\left.\begin{array}{l} \Psi_A^{M_\Sigma}(\psi_1, \beta, \psi_4\gamma_1) = \langle \psi_1\beta, \psi_4\gamma_1\rangle = \langle A_1\psi_1, \psi_4\gamma_1\rangle = -4A_1 \\ \Psi_l^N(-\phi_1 h, h^2, -8\phi_4\omega^2 h^2) = 16 \end{array}\right\} \implies A_1 = -4$$

Analogously $A_2 = -4$.

Regarding to the piece corresponding to the representation $\Lambda_0^2 H_3$, we know from the proof of [5, lemma 14] that $\psi_i\psi_j = \psi_i \cup \psi_j$, for any $1 \leq i, j \leq 6$. We have thus proven the following

Theorem 7. *For the Riemann surface Σ of genus $g = 3$, the isomorphism $H^*(M_\Sigma) \stackrel{\cong}{\to} QH^*(M_\Sigma)$ is given by*

$$\begin{cases} \alpha^2 = \alpha_2 \\ \alpha\beta = \alpha_1\beta_1 + 4\alpha \\ \beta^2 = \beta_2 + 16\beta - 8\alpha_2 \\ \alpha\gamma = \alpha_1\gamma_1 - 3\beta - 3\alpha_2 - 1 \\ \beta\gamma = \beta_1\gamma_1 - 20\gamma - 12\alpha_1\beta_1 + 24\alpha \\ \gamma^2 = \gamma_2 + 8\gamma_1\alpha_1 - 6\beta_2 - 24\alpha_2 - \beta - 8 \\ \psi_i\alpha = \psi_i\alpha_1, & 1 \leq i \leq 6 \\ \psi_i\beta = \psi_i\beta_1 - 4\psi_i, & 1 \leq i \leq 6 \\ \psi_i\gamma = \psi_i\gamma_1 - 4\psi_i\alpha_1, & 1 \leq i \leq 6 \\ \psi_i\psi_j = \psi_i \cup \psi_j, & 1 \leq i, j \leq 6 \end{cases}$$

5. Some computations of Gromov-Witten invariants of degree two

To finish it only remains to compute the following Gromov-Witten invariants of degree two: $\Psi_{2A}^{M_\Sigma}(\beta, \beta, \mathrm{pt})$ and $\Psi_{2A}^{M_\Sigma}(\alpha, \gamma, \mathrm{pt})$. Fix three points $P_0, P_1, P_2 \in \mathbb{P}^1$. Then $\Psi_{2A}^{M_\Sigma}(\alpha, \gamma, \mathrm{pt})$ equals

$$\#\{f : \mathbb{P}^1 \to M_\Sigma | f \text{ is holomorphic}, f_*[\mathbb{P}^1] = 2A, f(P_0) = \mathrm{pt}, f(P_1) \in V_\alpha, f(P_2) \in V_\gamma\},$$

and analogously for the other. Fix a generic point $\mathrm{pt} \in M_\Sigma$, corresponding to a bundle E on Σ. We recall the following result from [2, section 6] (here we suppose that the degree of the determinant $\Lambda = \det E$ is one).

Proposition 8. *Let $\mathcal{U} \to \Sigma \times M_\Sigma$ be the universal bundle. For any holomorphic map $f : \mathbb{P}^1 \to M_\Sigma$ such that $f_*[\mathbb{P}^1] = 2A$ put $\mathbb{E} = (1 \times f)^*\mathcal{U} \to \Sigma \times \mathbb{P}^1$. Denote by p and q the*

projections of $\Sigma \times \mathbb{P}^1$ *onto* Σ *and* $\mathbb{P}^1$ *respectively. Then one and only one of the following two cases hold*

- *There is a stable bundle* V *on* Σ *of rank* 2 *and degree zero and some* $x \in \Sigma$ *such that* $0 \to p^*V \otimes q^*\mathcal{O}(1) \to \mathbb{E} \to p^*\mathbb{C}(x) \to 0$.
- *There is a line bundle* L *on* Σ *of degree zero such that* $0 \to p^*L \otimes q^*\mathcal{O}(2) \to \mathbb{E} \to p^*L^{-1} \otimes \Lambda \to 0$.

Proposition 9. *Let*

$$\mathcal{R} = \{f : \mathbb{P}^1 \to M_\Sigma | f \text{ is holomorphic}, f_*[\mathbb{P}^1] = 2A, pt \in f(\mathbb{P}^1)\}.$$

endowed with the action of $PGL(2,\mathbb{C})$ *given by reparametrization of* $\mathbb{P}^1$. *Then* $\mathcal{R}$ *is a principal fibre bundle* $PGL(2,\mathbb{C}) \to \mathcal{R} \to \mathbb{P}(E)$. *So* $\mathcal{R}$ *is smooth, it has dimension* $\dim_{\mathbb{C}} \mathcal{R} = 5$ *and* $\mathbb{P}(E)$ *parametrizes "non-parametrized"* $\mathbb{P}^1 \to M_\Sigma$ *representing the homology class* $2A$ *and passing through* pt.

Proof. The second case of proposition 8 corresponds to the maps $f : \mathbb{P}^1 \to M_\Sigma$ whose image lies in the subspace of bundles defined by N described in section 4, which is of dimension 5. Therefore for a generic point, $\mathrm{pt} \notin N$ and hence any $f \in \mathcal{R}$ is in the first case of proposition 8.

Now let $f \in \mathcal{R}$ and put $\mathbb{E} = (1 \times f)^*\mathcal{U}$. There exists a stable bundle V on Σ of rank 2 and degree zero and some $x \in \Sigma$ such that

$$0 \to p^*V \otimes q^*\mathcal{O}(1) \to \mathbb{E} \to p^*\mathbb{C}(x) \to 0. \tag{4}$$

Let $P \in \mathbb{P}^1$ with $f(P) = \mathrm{pt}$, then restricting to $\Sigma \times \{P\}$ we get that $0 \to V \to E \to \mathbb{C}(x) \to 0$. So the possible V appearing for $f \in \mathcal{R}$ are given by quotients of E onto a skyscraper sheaf supported on a single point of Σ. This corresponds to the 2-dimensional space parametrized by $\mathbb{P}(E)$.

Moreover the exact sequence (4) is an elementary transformation which can be reversed as

$$0 \to \mathbb{E} \to p^*(V \otimes \mathcal{O}(x)) \otimes q^*\mathcal{O}(1) \to p^*(\mathbb{C}(x) \otimes \mathcal{O}(x)) \otimes q^*\mathcal{O}(2) \to 0. \tag{5}$$

So the possible bundles $\mathbb{E}$ over $\Sigma \times \mathbb{P}^1$ appearing in an exact sequence (4) are parametrized by epimorphisms $p^*(V \otimes \mathcal{O}(x)) \otimes q^*\mathcal{O}(1) \twoheadrightarrow p^*(\mathbb{C}(x) \otimes \mathcal{O}(x)) \otimes q^*\mathcal{O}(2)$, i.e. (restricting to $\{x\} \times \mathbb{P}^1 \subset \Sigma \times \mathbb{P}^1$) by maps $\mathcal{O}(1) \oplus \mathcal{O}(1) \twoheadrightarrow \mathcal{O}(2)$, or equivalently, by two linear forms in $H^0(\mathbb{P}^1, \mathcal{O}(1))$ with no common zero. Now note that $\mathrm{PGL}(2,\mathbb{C})$ acts freely and transitively on these elements, so the possible diagrams (5) with V fixed are parametrized by $\mathrm{PGL}(2,\mathbb{C})$. The statement now follows easily. $\square$

The image of $\mathcal{R}$ in M_Σ, $\{f(P) | f \in \mathcal{R}, P \in \mathbb{P}^1\}$, is described as follows. Put $R = \mathcal{R} \times_{\mathrm{PGL}(2,\mathbb{C})} \mathbb{P}^1$, so there is a fibration $\mathbb{P}^1 \to R \to \mathbb{P}(E)$ and a natural map $R \to M_\Sigma$ whose

image is $\{f(P)|f \in \mathcal{R}, P \in \mathbb{P}^1\}$. Now R is smooth, compact and of dimension 3. Let us denote α_R, β_R and γ_R for the pull-back of α, β and γ from M_Σ to R, respectively. We have the following

Lemma 10. *Let $\bar{f} \in H^2(\mathbb{P}(E))$ denote the class of the fibre of $\mathbb{P}(E) \to \Sigma$. Then $\langle \gamma_R, [R]\rangle = -12$ and $\beta_R/[\mathbb{P}^1] = -\bar{f}$, where $\mathbb{P}^1$ is the fibre of $R \to \mathbb{P}(E)$.*

Proof. The cohomology ring of $\mathbb{P}(E)$ is $H^*(\mathbb{P}(E)) = H^*(\Sigma)[h]/(h^2 - \Lambda h)$, where h is the hyperplane class of $\mathbb{P}(E)$, since $c_1(E) = \Lambda$. Now we construct the universal bundle $\mathcal{V}$ parametrizing the bundles V which are quotients of E onto a skyscraper sheaf supported at one point of Σ. We have an exact sequence

$$0 \to \mathcal{V} \to \pi_1^* E \otimes \mathcal{O}_{\mathbb{P}(E)}(-1) \to \mathcal{O}_\Delta \to 0$$

on $\Sigma \times \mathbb{P}(E)$, where $\pi_1 : \Sigma \times \mathbb{P}(E) \to \Sigma$ is the projection and $\Delta \subset \Sigma \times \Sigma$ is the diagonal divisor (we omit some pull-backs when there is no danger of confussion). The total Chern class of $\mathcal{V}$ is

$$c(\mathcal{V}) = (1-\Delta)(1 + \pi_1^*\Lambda - 2h) = 1 + (\pi_1^*\Lambda - 2h - \Delta) + (2h - \pi_1^*\Lambda)\Delta$$

Let us construct the space R. The fibre of $\mathcal{R} \to \mathbb{P}(E)$ over V is given by the isomorphism $\mathrm{Hom}(V \otimes \mathcal{O}(x), \mathbb{C}(x) \otimes \mathcal{O}(x)) \xrightarrow{\cong} H^0(\mathbb{P}^1, \mathcal{O}(1))$. The action of $\mathrm{PGL}(2,\mathbb{C})$ is trivial on the first space and standard on the second. So fibre of $R \to \mathbb{P}(E)$ over V is $\mathbb{P}(\mathrm{Hom}(V, \mathbb{C}(x))^\vee) = \mathbb{P}(V_x)$, where V_x is the fibre of V at x. Thus

$$R = \mathbb{P}(\pi_*(\mathcal{V}|_\Delta)), \tag{6}$$

for $\pi : \Sigma \times \mathbb{P}(E) \to \mathbb{P}(E)$. The Chern character of $\pi_*(\mathcal{V}|_\Delta)$ is by Riemann-Roch

$$\begin{aligned} \mathrm{ch}\,(\pi_*(\mathcal{V}|_\Delta)) &= \mathrm{ch}\,\left(\pi_!(\mathcal{V}|_\Delta)\right) = \pi_*\left(\mathrm{ch}\,(\mathcal{V}|_\Delta)\,\mathrm{Todd}\,T_\Sigma\right) = \\ &= \pi_*\left((2\Delta + (\pi_1^*\Lambda - 2h - 2\Delta)\Delta - \frac{1}{2}(\pi_1^*\Lambda - 2h)\Delta^2)(1 - \frac{1}{2}K)\right) = \\ &= 2 + \Lambda + K - 2h \end{aligned}$$

on $\mathbb{P}(E)$, where K is the canonical class of Σ and using that $\pi_*\Delta^2 = -K$. We deduce that the cohomology ring of R is $H^*(R) = H^*(\mathbb{P}(E))[k]/(k^2 = c_1 k - c_2)$, where k is the hyperplane class of (6) and

$$\begin{aligned} c_1 &= \Lambda + K - 2h \\ c_2 &= -2hK \end{aligned}$$

The universal bundle over $\Sigma \times R$ is given by

$$0 \to \mathcal{E} \to \mathcal{V} \otimes \mathcal{O}(\Delta)\mathcal{O}_R(-1) \to \mathcal{O}_\Delta \otimes \mathcal{O}(\Delta) \to 0,$$

from where we get that $c_1(\mathcal{E}) = \pi_1^*\Lambda - 2h - 2k$ and $c_2(\mathcal{E}) = (2h - \pi_1^*\Lambda + k)\Delta + 2hK + kK$. Let $\mathfrak{g}_{\mathcal{E}}$ be the associated $SO(3)$-bundle to $\mathcal{E}$. We have

$$p_1(\mathfrak{g}_{\mathcal{E}}) = 4c_2(\mathcal{E}) - c_1(\mathcal{E})^2 = 4\Delta(2h + k - \pi_1^*\Lambda).$$

Then $\mu_R : H_*(\Sigma) \to H^{4-*}(R)$ is $\mu_R(z) = -\frac{1}{4}p_1(\mathfrak{g}_{\mathcal{E}})/z$, so that

$$\left\{ \begin{array}{l} \alpha_R = 2\mu_R(\Sigma) = 2\Lambda - 4h - 2k \\ \beta_R = -4\mu_R(x) = \\ -(8h + k)f \gamma_R = -2\sum_{i=1}^{3} \mu_R(\gamma_i)\mu_R(\gamma_{3+i}) = -6(2h + k)^2 f = -12hkf \end{array} \right.$$

where $f \in H^2(R)$ is the class of the fibre of $R \to \Sigma$. Therefore $\langle \gamma_R, [R] \rangle = -12$ and $\beta_R/[\mathbb{P}^1] = -\bar{f}$ where $\bar{f} \in H^2(\mathbb{P}(E))$ is the class of the fibre of $\mathbb{P}(E) \to \Sigma$. $\square$

Now we may finish our computation. We have

$$\Psi_{2A}^{M_\Sigma}(\alpha, \gamma, \mathrm{pt}) = \alpha[2A]\langle \gamma_R, [R] \rangle = -24, \tag{7}$$

since any line in $\mathcal{R}$ cuts V_α in $\alpha[2A]$ points. Also $\beta_R/[\mathbb{P}^1] \in H^2(\mathbb{P}(E))$ is cohomology class dual to the subset of "non-parametrized" $\mathbb{P}^1 \to M_\Sigma$ (representing $2A$ and passing through pt) which lie in (a cycle Poincaré-dual to) β_R. Therefore

$$\Psi_{2A}^{M_\Sigma}(\beta, \beta, \mathrm{pt}) = \langle (\beta_R/[\mathbb{P}^1]) \cup (\beta_R/[\mathbb{P}^1]), [\mathbb{P}(E)] \rangle = 0 \tag{8}$$

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